About the Derivation of the Quasilinear Approximation in Plasma Physics

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Abstract This contribution, built on the companion paper [1], is focused on the different mathematical approaches available for the analysis of the quasilinear approximation in plasma physics.

1 Introduction and Notation

The origin of this contribution is the issue of the approximation of solutions of the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0\,,$$

where $f(t, x, v)$ is a probability density driven by a self-consistent potential, given in terms of this density by the Poisson equation

$$-\,\Delta\Phi(t, x) = \rho(t, x) = \int_{\mathbb{R}^d_v} f(t, x, v)dv - 1\,, \quad E(t, x) = -\nabla\Phi(t, x)\,,$$

in the domain $\mathbb{T}^d \times \mathbb{R}^d_v$, with $\mathbb{T}^d = (\mathbb{R}_x/2\pi\mathbb{Z})^d$, by a parabolic (linear or nonlinear) diffusion equation for the space averaged density of particles, namely

$$\partial_t \overline{f}(t, v) - \nabla_v \cdot \big(\mathbb{D}(t, v)\nabla_v \overline{f}(t, v)\big) = 0\,. \tag{1}$$

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Such equation carries the name of quasilinear approximation and is a very active subject of plasma physics. Here, relying on a companion paper [1] (devoted to a more detailed physical analysis and more focused on the interpretation of turbulence), we focus on the different mathematical approaches motivated by all the recent progress (for instance, around the question of Landau damping) on the analysis of the Vlasov equation.

Starting from the natural scaling derived for instance in [1], we propose a rescaled version of the Vlasov equation and first show obstructions to the convergence to an equation of the type (1) with a non-zero diffusion. This leads to the introduction of a stochastic vector field, hence to a non- self-consistent Liouville equation. There, a direct approach (as in the contributions of A. Vasseur and coworkers [26, 28]) produces a complete positive answer.

A more classical analysis leads to the comparison with the present results on Landau damping and to an alternate approach based on the spectral theory, and at variance with the rescaled equation, valid only for short time. No complete proof is given, but a natural road for convergence based on some plasma physics computations is proposed. The complete proof will be addressed in a future work.

As a short-time correction, it may play in the subject the same role as was done by the introduction of the diffusion in the macroscopic limit of the Boltzmann equation by Ellis and Pinsky [13].

As a mathematical contribution to physics (however modest it may be), we dedicate this chapter to the memory of Alex Grossman. Besides being recognized for super scientific achievement in particular with the introduction of wavelets, he will be remembered over many years with his generous and charismatic influence on our community.

1.1 Notation and Some Hypotheses

The flow $S(t) : f \mapsto f(t, x - vt, v)$, with $s \mapsto x - vs$ denoting the free advection flow modulo $(2\pi)^d$ on $\mathbb{T}^d$, is the advection flow generated by the operator $-v \cdot \nabla_x$. In the same way, we also introduce the flow S^ε generated by the operator $-\varepsilon^{-2} v \cdot \nabla_x$ and defined by

$$S_t^\varepsilon f = S\left(\frac{t}{\varepsilon^2}\right) f = f\Big(t, x - v\frac{t}{\varepsilon^2}, v\Big) .$$

These are unitary groups in any $L^p(\mathbb{T}^d \times \mathbb{R}_v^+)$, with $1 \le p \le \infty$, which preserve the positivity and the total mass. Since this is not the relevant issue for our discussion, in the presence of an $\varepsilon > 0$, the initial data $f^\varepsilon(0, x, v)$ is assumed to be independent of ε and as smooth as required (hence, taking in account existing regularity results [16] for the Vlasov equation) to have global-in-time solutions that will satisfy the relevant computations. On the other hand, emphasis has to be put on the regularity estimates that are independent of ε .

As observed in many previous publications starting from Landford [23], limit can be obtained not at the level of the equation but at the level of the solution itself. As a consequence, we will use the first-order Duhamel formula

$$f(t,x,v) = S_t f(0) - \int_0^t d\sigma_1\, S_{t-\sigma_1} E(\sigma_1)\cdot\nabla_v f(\sigma_1)\,,$$

and in Sect. 2 an avatar of the second-order Duhamel, to connect the value of $f(\sigma_1)$ with the value of $f(\sigma_2)$ according to the Duhamel formula

$$f(\sigma_1) = S_{\sigma_1-\sigma_2} f(\sigma_2) - \int_{\sigma_2}^{\sigma_1} d\sigma\, S_{\sigma_1-\sigma} E(\sigma)\cdot\nabla_v f(\sigma)\,,$$

which finally gives

$$\begin{aligned} f(t) &= S_t f(0) - \int_0^t d\sigma_1\, S_{t-\sigma_1} E(\sigma_1)\cdot\nabla_v S_{\sigma_1-\sigma_2} f(\sigma_2) \\ &+ \int_0^t d\sigma_1 \int_{\sigma_2}^{\sigma_1} d\sigma\, S_{t-\sigma_1} E(\sigma_1)\cdot\nabla_v (S_{\sigma_1-\sigma} E(\sigma)\cdot\nabla_v f(\sigma))\,. \end{aligned}$$

Denoting by $\fint dx$ the x-average on $\mathbb{T}^d$, one obtains

$$\partial_t \fint f(t,x,v)dx + \nabla_v\cdot \fint E(t,x) f(t,x,v)dx = 0\,. \tag{2}$$

The second term of (2) is the divergence of the averaged flux

$$J = \fint E(t,x) f(t,x,v)dx\,.$$

Therefore, almost all the rest of this contribution is devoted to the determination of such flux (sometimes called a "Fick law"). Since weak convergence involves distributions and test functions, such duality is denoted by the following bracket notation $\langle .,.\rangle$. Using the fact that $\mathcal{D}(\mathbb{R}_t\times\mathbb{T}^d\times\mathbb{R}^d_v) = \overline{\mathcal{D}(\mathbb{R}_t)\otimes\mathcal{D}(\mathbb{T}^d)\otimes\mathcal{D}(\mathbb{R}^d_v)}$, test functions depending on one or an other of these spaces will be used according to convenience. The symbol $\overline{T^\varepsilon}$ will be used to denote any cluster point (in the sense of distributions or under some other stronger topology) of a family $\{T^\varepsilon\}_\varepsilon$ of bounded distributions. Scalar product in $\mathbb{R}^d$ is either denoted by the dot symbol $\cdot$ or enclosed in the parentheses $(\cdot,\cdot)$. Eventually, $z^\star$ denotes the complex conjugate of the complex number z.

1.2 The Rescaled Liouville Equation

Both from plasma physics considerations (see [1]) and also because it is compatible with the scaling invariance of the diffusion in the velocity variable (see Eq. (1)), densities $f^\varepsilon(t,x,v)$, solutions of the following rescaled (with $\varepsilon > 0$) Liouville equation,

$$\partial_t f^\varepsilon + \frac{v}{\varepsilon^2}\cdot\nabla_x f^\varepsilon + \frac{E^\varepsilon}{\varepsilon}\cdot\nabla_v f^\epsilon = 0\,,\quad \text{with}\quad E^\varepsilon = -\nabla\Phi^\varepsilon\,, \tag{3}$$

are considered.

As it will soon appear below, the specific behavior of the solutions as $\varepsilon \to 0$ (and/or as $t \to \infty$) depends on the time regularity of the potential rather than the space regularity and on properties of the initial data. Then, unless otherwise specified, it is assumed below that the initial data $f_0 \in \mathcal{S}(\mathbb{T}^d \times \mathbb{R}^d_v)$ is an ε-independent smooth function and that $E^\varepsilon = -\nabla\Phi^\varepsilon$ is, locally in time, uniformly Lipschitz with respect to the variable x. Under such hypotheses, solutions of Eq. (3) are well and uniquely defined. On the other hand, no assumption is made on the uniform regularity of the solution $f^\varepsilon(t,x,v)$ either as $t \to \infty$ or as $\varepsilon \to 0$.

Hence, only ε-independent estimates, which are in agreement with the classical results including in particular those concerning the solution of the Vlasov equation, are based on the fact that the Liouville equation preserves positivity and Lebesgue measure.

$$\forall t \in \mathbb{R}^+,\quad 0 \le f^\varepsilon(t,x,v) \le \sup_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d_v} f_0(x,v),\quad \int_{\mathbb{T}^d\times\mathbb{R}^d_v} f^\varepsilon(t,x,v)dxdv = 1\,,$$

$$\forall 1 \le p \le \infty,\quad \forall t \in \mathbb{R}^+,\quad \|f^\varepsilon(t)\|_{L^p(\mathbb{T}^d\times\mathbb{R}^d_v)} = \|f(0)\|_{L^p(\mathbb{T}^d\times\mathbb{R}^d_v)}\,.$$

To use the scaling and the ergodicity of the d-dimensional torus, the following proposition is recalled.

Proposition 1 (Ergodicity)

1. *Any $g \in L^p(\mathbb{T}^d \times \mathbb{R}^d_v)$, with $1 \le p \le \infty$, which satisfies the relation*

$$v\cdot\nabla_x g = 0,\quad in\quad \mathcal{D}'(\mathbb{T}^d\times\mathbb{R}^d_v)\,, \tag{4}$$

is an x-independent function.

2. *The solutions f^ε of the equation*

$$\varepsilon^2\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon = 0\,,\quad f^\varepsilon(0,x,v) = f_0(x,v) \in L^p(\mathbb{T}^d\times\mathbb{R}^d_v)\,, \tag{5}$$

converge in $L^\infty(\mathbb{R}^+_t;\mathcal{D}'(\mathbb{R}^d_v))$ to $\fint f_0(x,v)dx$.

This proposition and its use in Landau damping are classical (see [5, 18]), and, for sake of completeness, its proof is shortly recalled below.

Proof From (4), one deduces the relation

$$\frac{d}{dt}g(x-vt,v)=0\,,$$

which by Fourier transform gives

$$\big(1-\exp(ik\cdot vt)\big)\hat{g}(k,v)=0,\quad \forall k\in\mathbb{Z}^d,\ v\in\mathbb{R}^d,\ t\in\mathbb{R}\,.$$

This relation implies that the support of $\hat{g}$ is contained in the set

$$\mathrm{supp}(\hat{g}):=\{(k,v)\in\mathbb{Z}^d\times\mathbb{R}^d\mid k\cdot vt\in 2\pi\mathbb{Z},\ \forall t\in\mathbb{R}\}\,.$$

For any $\delta,\ T,\ r,\ R>0$, such that $\delta<T$ and $r<R$, the Lebesgue measure of the set $\mathrm{supp}(\hat{g})$ for $\delta<t<T$ and $r<|v|<R$ is zero. Therefore, since g belongs to $L^p(\mathbb{T}^d\times\mathbb{R}^d_v)$, this forces $\hat{g}$ to be equal to zero for all $k\neq 0$, and finally, one obtains

$$g=\hat{g}(0,v)=\fint g(x,v)dx\,.$$

In the same way for the point 2, use the fact that the solution of (5) is given by $f^\varepsilon(t,x,v)=f_0(x-vt/\varepsilon^2,v)$ to write, for all $k\in\mathbb{Z}^d$ and for any $\phi\in\mathcal{D}(\mathbb{R}^d_v)$,

$$\begin{aligned}\int_{\mathbb{R}^d_v}\hat{f}^\varepsilon(t,k,v)\phi(v)dv &=\int_{\mathbb{R}^d_v}dv\fint dx\, f^\varepsilon(t,x,v)e^{-ik\cdot x}\phi(v)dv\\ &=\int_{\mathbb{R}^d_v}\hat{f}_0(k,v)\phi(v)e^{ik\cdot v\frac{t}{\varepsilon^2}}dv\,.\end{aligned}\tag{6}$$

Since $f_0(x,v)\in L^p(\mathbb{T}^d\times\mathbb{R}^d_v)$, by the Riemann– Lebesgue theorem, the right-hand side of (6) goes to 0 for $k\neq 0$ as $\varepsilon\to 0$, hence completing the proof of the point 2. □

As a consequence, writing the rescaled Liouville equation in the following form,

$$v\cdot\nabla_x f^\varepsilon=-\varepsilon^2\partial_t f^\varepsilon-\varepsilon E^\varepsilon\cdot\nabla_v f^\varepsilon,\tag{7}$$

one deduces from the uniform estimates that any cluster point $\overline{f^\varepsilon}=\overline{f^\varepsilon}(t,v)$ of the family $\{f^\varepsilon\}_\varepsilon$, in the $L^\infty(\mathbb{R}^+_t\times\mathbb{T}^d\times\mathbb{R}^d_v)$ weak$-\star$ topology, is independent of x and is a solution of the equation

$$\partial_t\overline{f^\varepsilon}+\nabla_v\cdot\left(\overline{\fint\frac{E^\varepsilon f^\varepsilon}{\varepsilon}dx}\right)=0\,.\tag{8}$$

In (8), the Fick law (relating the variation of the density to the divergence of the current) appears as the ratio of two terms going **at least formally** to zero because under the hypothesis $\overline{E^\varepsilon f^\varepsilon} = \overline{E^\varepsilon}\,\overline{f^\varepsilon}$ one has

$$\overline{\fint E^\varepsilon f^\varepsilon dx} = -\fint \nabla\overline{\Phi^\varepsilon}(t,x)\overline{f^\varepsilon}(t,v)dx = 0\,.$$

The justification of the quasilinear approximation would be the proof that

$$\overline{\fint \frac{E^\varepsilon f^\varepsilon}{\varepsilon} dx} = -\mathbb{D}(t,v)\nabla_v \overline{f^\varepsilon}\,, \tag{9}$$

with $\mathbb{D}(t,v)$ being a non-negative diffusion matrix.

1.3 Obstruction to the Convergence to a Non-degenerate Diffusion Matrix

The Liouville equation is the paradigm of an Hamiltonian system, while the diffusion equation is the model of an irreversible phenomenon. Then a paradox comes from the comparison of the two equations

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d_v} dv \fint dx\, |f^\varepsilon|^2 = 0 \quad \text{and}\\
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d_v} dv \fint dx\, |\overline{f^\varepsilon}|^2 + \int_{\mathbb{R}^d_v} dv \fint dx \left(\mathbb{D}(t,v)\nabla_v\overline{f^\varepsilon}\,,\ \nabla_v\overline{f^\varepsilon}\right) = 0\,.
\end{aligned} \tag{10}$$

This paradox has to be resolved to justify, in some cases, that the diffusion matrix $\mathbb{D}$ is not degenerate, taking in account for instance the following:

Proposition 2 *Assuming that $E^\varepsilon = -\nabla\Phi^\varepsilon$ is uniformly bounded in $L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))$ and that $f^\varepsilon(t,v)$ is the solution of (7) with $\overline{f^\varepsilon(0,x,v)} = \fint f_0^\varepsilon(x,v)dx$, one has the following facts:*

1. *The density $f^\varepsilon(t,x,v)$ converges strongly in $L^2([0,T]\times\mathbb{T}^d\times\mathbb{R}^d_v)$ if and only if*

$$\int_0^T dt \int_{\mathbb{R}^d_v} dv \left(\mathbb{D}(t,v)\nabla_v\overline{f^\varepsilon}(t,v)\,,\ \nabla_v\overline{f^\varepsilon}(t,v)\right) = 0\,.$$

2. *If $\partial_t\Phi^\varepsilon(t,x)$ is bounded in some distribution space $L^1(0,T;H^{-\beta}(\mathbb{T}^d))$ with some β finite and if $\varepsilon\partial_t\Phi^\varepsilon$ converges to 0 in $L^1([0,T]\times\mathbb{T}^d)$, then one obtains*

$$\partial_t\overline{f^\varepsilon} = 0 \quad \text{and} \quad v\cdot\overline{\fint \frac{E^\varepsilon f^\varepsilon}{\varepsilon}dx} = 0 \quad \text{on} \quad [0,T]\times\mathbb{R}^d_v\,.$$

Proof The point 1 is a direct consequence of the comparison between the classical Hilbertian estimate

$$\forall t \in (0,T), \qquad \int_0^t \overline{\|f^\varepsilon(s)\|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d_v)}}ds \geq \int_0^t \| \overline{f^\varepsilon(s)} \|^2_{L^2(\mathbb{T}^d\times\mathbb{R}^d_v)}ds$$

and the two equations appearing in the formula (10). For the point 2, multiply the rescaled Liouville equation by $\varepsilon\Phi^\varepsilon(t,x)\theta(t)\phi(v)$ to obtain after integration

$$\begin{aligned}
&\int_{\mathbb{R}^+_t}\int_{\mathbb{R}^d_v}\theta(t)\phi(v) ⨍ \frac{v\cdot\nabla_x f^\varepsilon(t,x,v)\Phi^\varepsilon(t,x)dx}{\varepsilon}dvdt\\
&=\int_{\mathbb{R}^+_t}\int_{\mathbb{R}^d_v}\theta(t)\nabla_v\phi(v)\cdot ⨍ E^\varepsilon(t,x)\Phi^\varepsilon(t,x)f^\varepsilon(t,x,v)dxdvdt \qquad (11)\\
&+\varepsilon\int_{\mathbb{R}^+_t}dt ⨍ dx\int_{\mathbb{R}^d_v}dv\, f^\varepsilon(t,x,v)\partial_t(\theta(t)\Phi^\varepsilon(t,x))\phi(v)\,.
\end{aligned}$$

Now, since $\varepsilon\partial_t\Phi^\varepsilon(t,x)$ converges strongly to zero in $L^1([0,T]\times\mathbb{T}^d)$ as $\varepsilon\to 0$, the last term of the right-hand side of (11) goes to 0 as $\varepsilon\to 0$, while for the first term of the right-hand side of (11) with the Aubin–Lions theorem (see for instance [30]), one obtains

$$\begin{aligned}
&\overline{\int_{\mathbb{R}^+_t}\int_{\mathbb{R}^d_v}\theta(t)\nabla_v\phi(v)\cdot ⨍ E^\varepsilon(t,x)\Phi^\varepsilon(t,x)f^\varepsilon(t,x,v)dxdvdt}\\
&=-\int_{\mathbb{R}^+_t}\int_{\mathbb{R}^d_v}\theta(t)\nabla_v\phi(v)\cdot ⨍ \frac{1}{2}\overline{\nabla_x|\Phi^\varepsilon(t,x)|^2}\,\overline{f^\varepsilon(t,x,v)}dxdvdt \qquad (12)\\
&=-\int_{\mathbb{R}^+_t}\int_{\mathbb{R}^d_v}\theta(t)\overline{f^\varepsilon}(t,v)\nabla_v\phi(v)\cdot ⨍ \frac{1}{2}\nabla_x\overline{|\Phi^\varepsilon(t,x)|^2}dx=0\,.
\end{aligned}$$

Eventually, using $E^\varepsilon=-\nabla\Phi^\varepsilon$ and an integration by parts in the variable x, for the left-hand side of (9), we obtain from (11)–(12)

$$\begin{aligned}
&\overline{\int_{\mathbb{R}^+_t}\int_{\mathbb{R}^d_v}\theta(t)\phi(v)v\cdot ⨍ \frac{E^\varepsilon(t,x)f^\varepsilon(t,x,v)}{\varepsilon}dxdvdt}\\
&=-\overline{\int_{\mathbb{R}^+_t}\int_{\mathbb{R}^d_v}\theta(t)\phi(v) ⨍ \frac{v\cdot\nabla_x\Phi^\varepsilon(t,x)f^\varepsilon(t,x,v)}{\varepsilon}dxdvdt} \qquad (13)\\
&=\overline{\int_{\mathbb{R}^+_t}\int_{\mathbb{R}^d_v}\theta(t)\phi(v) ⨍ \frac{v\cdot\nabla_x f^\varepsilon(t,x,v)\Phi^\varepsilon(t,x)}{\varepsilon}dxdvdt}=0\,.
\end{aligned}$$

Then, one observes that any vector-valued function $v\mapsto\psi(v)\in\mathcal{D}(\mathbb{R}^d_v;\mathbb{R}^d_v)$, with $\psi(0)=0$, can be written with the introduction of a function $v\mapsto\gamma(v)\in\mathcal{D}(\mathbb{R}^d_v)$

equal to 1 on the support of $\psi(v)$ as

$$\psi(v) = \left(\int_0^1 \nabla_v \psi(sv)ds\right) v \quad = \gamma(v)\left(\int_0^1 \nabla_v \psi(sv)ds\right) v = \varphi(v)v, \qquad \text{with } \; \varphi(v) := \gamma(v)\int_0^1 \nabla_v \psi(sv)ds\,. \tag{14}$$

From (13)–(14), one concludes that for any such vector-valued function $v \mapsto \psi(v)$ with $\psi(0) = 0$, one obtains

$$\int_{\mathbb{R}_t^+} \theta(t) \int_{\mathbb{R}_v^d} \psi(v) \cdot \overline{\fint \frac{E^\varepsilon(t,x) f^\varepsilon(t,x,v)}{\varepsilon} dx} dv dt = 0\,.$$

Therefore, the support of $\partial_t \overline{f^\varepsilon}(t,v)$ is contained in $[0,T] \times \{v = 0\}$. Hence, for any $\theta(t) \in \mathcal{D}(\mathbb{R}_t^+)$ and for any $\phi(v) \in \mathcal{D}(\mathbb{R}_v^d)$ with the point 0 not included in the support of $\phi(v)$, one obtains

$$\int_{\mathbb{R}_t^+ \times \mathbb{R}_v^d} \overline{\fint f^\varepsilon dx} \partial_t \theta(t) \phi(v) dv dt = 0\,. \tag{15}$$

However, since $\overline{\fint f^\varepsilon dx} \in L^\infty(\mathbb{R}_t^+ \times \mathbb{R}_v^d)$, relation (15) remains valid for any test function $\phi \in \mathcal{D}(\mathbb{R}_v^d)$, and this completes the proof of the point 2. ◻

Remark 1 For Vlasov–Poisson equations, the "ergodic" convergence of $f^\varepsilon(t,x,v)$ to $\overline{f^\varepsilon(t,v)}$, which has already been proven, implies (using the Poisson equation) the weak convergence to zero of the electric field $E^\varepsilon(t,x)$. This is in this scaling the "baby Landau damping." Strong convergence will be equivalent to the genuine Landau damping as proven in Theorem 3 of Sect. 3.3. As a consequence, too much time regularity for the electric field E^ε may prevent a limit described by a non-degenerate diffusion equation.

Remark 2 From the above observations, one concludes that, as it is the case in many related examples, proofs should rather involve the behavior of the solution itself rather than the asymptotic structure of the equation. Therefore, the use of the Duhamel expansion (up to convenient order) appears to be a natural tool, and it will appear in two very different approaches. The first one is based on the introduction of stochasticity in the electric field (see Sect. 2). Hence, it corresponds to a situation where such electric field is non-self-consistent: it is a genuine Liouville equation. The second approach, based on a short-time asymptotic (see Sect. 3.4), deals with configurations where the electric field is given self-consistently (i.e., determined from the density of particles) through a spectral analysis.

1.4 The First Iteration of the Duhamel Formula and the Diffusion: Reynolds Electric Stress Tensor

From the previous section, one deduces that the convergence to a genuine diffusion equation requires that the vector field E^ε or the potential Φ^ε becomes "turbulent" as $\varepsilon \to 0$. This justifies at present the introduction of the diffusion tensor $\mathbb{D}^\varepsilon$. Using the Duhamel formula

$$f^\varepsilon(t) = S_t^\varepsilon f_0^\varepsilon - \frac{1}{\varepsilon}\int_0^t S_{t-\sigma}^\varepsilon E^\varepsilon(\sigma)\cdot\nabla_v f^\varepsilon(\sigma)d\sigma\,, \tag{16}$$

and the ergodicity (see Proposition 2), the first term of the right-hand side of (16) is ignored, while multiplying by a test function $\phi(v)$, one obtains for the Fick term the following expression:

$$\begin{aligned}&-\overline{\fint dx\int_{\mathbb{R}_v^d} dv\,\phi(v)\nabla_v\cdot\left(\frac{E^\varepsilon f^\varepsilon}{\varepsilon}\right)} =\\ &-\frac{1}{\varepsilon^2}\overline{\int_{\mathbb{R}_v^d} dv\,\nabla_v\phi(v)\cdot\int_0^t ds\fint dx\,E^\varepsilon(t)S_{t-s}^\varepsilon(E^\varepsilon(s)\cdot\nabla_v f^\varepsilon(s))}\,.\end{aligned} \tag{17}$$

Using the explicit formula,

$$S_{t-s}^\varepsilon(E^\varepsilon(s)\cdot\nabla_v f^\varepsilon(s)) = E^\varepsilon(s,x-v(t-s)/\varepsilon^2)\cdot(\nabla_v f^\varepsilon)(s,x-v(t-s)/\varepsilon^2,v)\,, \tag{18}$$

the change of variable $\sigma = (t-s)/\varepsilon^2$, and the 2π-periodicity of the functions E^ε and f^ε, one obtains from (17)

$$\begin{aligned}&-\overline{\fint dx\int_{\mathbb{R}_v^d} dv\,\phi(v)\nabla_v\cdot\left(\frac{E^\varepsilon f^\varepsilon}{\varepsilon}\right)} =\\ &-\overline{\int_{\mathbb{R}_v^d} dv\,(\nabla_v\phi(v))^T\int_0^{\frac{t}{\varepsilon^2}} d\sigma\fint dx\,E^\varepsilon(t,x+\sigma v)\otimes E^\varepsilon(t-\varepsilon^2\sigma,x)\nabla_v f^\varepsilon(t-\sigma\varepsilon^2,x,v)}\\ &=\\ &\overline{\int_{\mathbb{R}_v^d} dv\fint dx\int_0^{\frac{t}{\varepsilon^2}} d\sigma f^\varepsilon(t-\sigma\varepsilon^2,x,v)\nabla_v\cdot\big(E^\varepsilon(t-\varepsilon^2\sigma,x)\otimes E^\varepsilon(t,x+\sigma v)\nabla_v\phi(v)\big)}\,.\end{aligned} \tag{19}$$

Observe that the above integrations by part are justified on the following ground:

(i) All the arguments, E^ε and f^ε, are assumed for $\varepsilon > 0$ to be smooth enough.
(ii) The smooth test function ϕ is independent of x and t.

Further analysis may be decomposed into four steps:

1. In (19), replacing f^ε by an other smooth test function, one introduces the diffusion tensor $\mathbb{D}^\varepsilon(t,v)$ defined by

$$\int_{\mathbb{R}^d_v} dv\,\big(\mathbb{D}^\varepsilon(t,v)\nabla_v\psi(v)\,,\ \nabla_v\phi(v)\big) = \int_{\mathbb{R}^d_v} dv\,\big(\nabla_v\psi(v)\,,\ \mathbb{D}^\varepsilon(t,v)^T\nabla_v\phi(v)\big)$$
$$= \int_{\mathbb{R}^d_v} dv\,(\nabla_v\psi(v))^T\Big(\fint dx\int_0^{\frac{t}{\varepsilon^2}} d\sigma\ E^\varepsilon(t-\varepsilon^2\sigma,x)\otimes E^\varepsilon(t,x+\sigma v)\Big)\nabla_v\phi(v)\,.$$

2. Eventually observe that under a decorrelation hypothesis and with

$$\overline{f^\varepsilon(t-\sigma\varepsilon^2,x,v)} = \overline{f^\varepsilon}(t,v)\,,$$

one would obtain

$$\overline{\int_{\mathbb{R}^d_v} dv\fint dx\int_0^{\frac{t}{\varepsilon^2}} d\sigma f^\varepsilon(t-\sigma\varepsilon^2,x,v)\nabla_v\cdot\big(E^\varepsilon(t-\varepsilon^2\sigma,x)\otimes E^\varepsilon(t,x+\sigma v)\nabla_v\phi(v)\big)}$$
$$= \int_{\mathbb{R}^d_v}\overline{f^\varepsilon}(t,v)\nabla_v\cdot\big(\overline{\mathbb{D}^\varepsilon(t,v)^T}\nabla_v\phi(v)\big)dv\,.$$

3. With the notation

$$E^\varepsilon(t-\varepsilon^2\sigma,x)\otimes E^\varepsilon(t,x+\sigma v) = E(T,X)\otimes E(S,Y)_{T=t-\varepsilon^2\sigma,\,X=x,\,S=t,\,Y=x+\sigma v}\,,$$

appears some type of average of the tensor that in the present field ("plasma turbulence") plays a role very similar to the Reynolds hydrodynamic stress tensor $u(t,x)\otimes u(s,y)$ in fluid turbulence.

4. Since the electric field is the gradient of a real x-periodic potential, i.e., $E^\varepsilon(t,x) = -\nabla\Phi^\varepsilon(t,x)$, with the notation

$$E^\varepsilon(t,k) = \fint E^\varepsilon(t,x)e^{-ik\cdot x}dx \quad\text{and}\quad \Phi^\varepsilon(t,k) = \fint \Phi(t,x)e^{-ik\cdot x}dx\,,$$

one has

$$\mathbb{D}^\varepsilon(t,v) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} k\otimes k\int_0^{\frac{t}{\varepsilon^2}}\Phi^\varepsilon(t-\varepsilon^2\sigma,k)(\Phi^\varepsilon(t,k))^\star e^{-ik\cdot v\sigma}d\sigma\,,$$

with

$$\sum_{k\in\mathbb{Z}^d}|k|^4|\Phi^\varepsilon(t,k)|^2 \le C\ \text{ independent of } \varepsilon\,.$$

In order to compare deterministic results of this section with the stochastic ones of Sect. 2, we state and prove the following:

Proposition 3 *We assume that the Fourier coefficients of the potential* $\Phi^\varepsilon(t,x)$ *are given by the ansatz*

$$\Phi^\varepsilon(t,k) = \underline{\Phi}^\varepsilon(t,k)e^{-i\omega(k)\varepsilon^{-\beta_k}t}\,,$$

where $\omega(k)\varepsilon^{-\beta_k}$ *is a fast time frequency with* $\omega(-k)=\omega(k)$ *and* $\beta_{-k}=\beta_k$*, while the amplitude* $\underline{\Phi}^\varepsilon(t,k)$ *is slowly varying with time and more precisely satisfies the estimate*

$$\sum_{k\in\mathbb{Z}^d} |k|^4|\underline{\Phi}^\varepsilon(t,k)|^2 \le C \;\; \textit{independent of}\; \varepsilon\,. \tag{20}$$

Then:

1. *The diffusion tensor* $\mathbb{D}^\varepsilon$ *is given by*

$$\begin{aligned}\mathbb{D}^\varepsilon(t,v) &= \fint dx\int_0^{\frac{t}{\varepsilon^2}} d\sigma\; E^\varepsilon(t,x+\sigma v)\otimes E^\varepsilon(t-\varepsilon^2\sigma,x)\\ &= \sum_{k\in\mathbb{Z}^d} k\otimes k \int_0^{\frac{t}{\varepsilon^2}} d\sigma\; |\underline{\Phi}^\varepsilon(t,k)|^2 e^{-i(\varepsilon^{2-\beta_k}\omega(k)-k\cdot v)\sigma}\\ &= \sum_{k\in\mathbb{Z}^d} k\otimes k\; |\underline{\Phi}^\varepsilon(t,k)|^2\, \frac{\sin\left((\varepsilon^{2-\beta_k}\omega(k)-k\cdot v)\frac{t}{\varepsilon^2}\right)}{\varepsilon^{2-\beta_k}\omega(k)-k\cdot v}\,.\end{aligned}$$

2. *Using the definition of the following hyperplanes,*

$$\pi_k^0 = \{v\in\mathbb{R}_v^d \;\; \textit{such that}\;\; k\cdot v = 0\}\,,$$

and

$$\pi_k^1 = \left\{v\in\mathbb{R}_v^d \;\; \textit{such that}\;\; \omega(k)-k\cdot v=0 \;\; or \;\; k\cdot(\boldsymbol{\omega}(k)-v)=0 \;\; with \;\; \boldsymbol{\omega}(k) := \frac{\omega(k)}{|k|}\frac{k}{|k|}\right\}\,,$$

for any $\psi,\ \phi\in\mathcal{D}(\mathbb{R}_v^d)$*, one obtains the following behavior as* $\varepsilon\to 0$*:*

$$\begin{aligned}&\int_{\mathbb{R}_v^d} (\nabla_v\phi(v))^T\,\overline{\mathbb{D}^\varepsilon(t,v)}\nabla_v\psi(v)\,dv = \pi\sum_{k\neq 0,\,\beta_k<2} \overline{|\underline{\Phi}_k^\varepsilon|^2}\int_{\pi_k^0} k\cdot\nabla_v\phi(v)\,k\cdot\nabla_v\psi(v)\,dv\\ &+\pi\sum_{k\neq 0,\,\beta_k=2} \overline{|\underline{\Phi}_k^\varepsilon|^2}\int_{\pi_k^1} k\cdot\nabla_v\phi(v)\,k\cdot\nabla_v\psi(v)\,dv\,.\end{aligned}$$

Proof As above, the proof uses the Fourier expansion of the vector field $E^\varepsilon(t,x) = -\nabla\Phi^\varepsilon(t,x)$, the limit in the sense of distribution of the function $\sin(s/\varepsilon)$, and finally the fact that for $\beta_k > 2$ one obtains

$$\lim_{\varepsilon\to 0} \frac{\sin\left((\varepsilon^{2-\beta_k}\omega(k) - k\cdot v)\frac{t}{\varepsilon^2}\right)}{\varepsilon^{2-\beta_k}\omega(k) - k\cdot v} = 0\,.$$

Remark 3 In point 2 of Proposition 3, one observes that under the decreasing condition (20), the cluster point $\overline{\mathbb{D}^\varepsilon}$ is the distribution

$$\overline{\mathbb{D}^\varepsilon} = \pi \sum_{k\in\mathbb{Z}^d,\,\beta_k<2} k\otimes k\, \overline{|\underline{\Phi}^\varepsilon(t,k)|^2}\,\delta(k\cdot v) + \pi \sum_{k\in\mathbb{Z}^d,\,\beta_k=2} k\otimes k\, \overline{|\underline{\Phi}^\varepsilon(t,k)|^2}\,\delta(\omega(k)-k\cdot v).$$

In the next section, the introduction of stochasticity in the vector field (or in the potential) has in particular the effect of smoothing the diffusion kernel by the introduction of regularized densities of

$$\delta(\omega(k) - k\cdot v) = \delta(k\cdot(\boldsymbol{\omega} - v)),$$

as it appears in Section 4.4 of [1] for the "resonance broadening"—like approximation of the "phase velocity" $\omega(k)/|k|$ (in terms of amplitude) or $\boldsymbol{\omega}(k)$ (in terms of vector).

This is, besides physical observations concerning turbulence effects in plasma, a complete justification to consider in the rescaled Liouville equation stochastic vector fields or potentials as it is done in the next section.

2 The Non-self-consistent Stochastic Approach

Following the above remark, in this section, we consider situations where vector field $E^\varepsilon = -\nabla\Phi^\varepsilon$ (or its potential) is a random variable as such being defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$ a σ-finite probability measure. The expectation of a random variable f is given by

$$\mathbb{E}[f] = \int_\Omega f(\omega)d\mathbb{P}(\omega)\,. \tag{21}$$

First standard hypotheses well adapted to our presentation are assumed on the vector field E^ε:

- **H1**. Stochastic average of E^ε set equal to zero, i.e.,

$$\forall\,(t,x), \quad \mathbb{E}[E^\varepsilon(t,x)] = 0\,. \tag{22}$$

- **H2**. Finite-time decorrelation: There exists a finite positive number τ such that

$$|t-s| \geq \tau\varepsilon^2 \implies \mathbb{E}[E^\varepsilon(t,x) \otimes E^\varepsilon(s,y)] = 0, \quad \forall (x,y)\,. \tag{23}$$

- **H3**. Time and space homogeneity. To emphasize the interplay between time oscillations and randomness, one assumes, for the Fourier coefficients of the vector field (then also for the potential), the following form:

$$E^\varepsilon(t,k) = \fint E^\varepsilon(t,x)e^{-ik\cdot x}dx = \underline{E}^\varepsilon(t,k)e^{-i\frac{\omega_k t}{\varepsilon^2}} = -ik\underline{\Phi}^\varepsilon(t,k)e^{-i\frac{\omega_k t}{\varepsilon^2}}\,. \tag{24}$$

Potentials $\underline{\Phi}^\varepsilon(t,k)$ are complex random variables, while frequencies ω_k are real and $(t,\ \varepsilon)$-independent. We also have the following parity properties:

$$\forall k \in \mathbb{Z}^d, \quad \underline{\Phi}^\varepsilon(t,-k) = (\underline{\Phi}^\varepsilon(t,k))^\star \quad \text{and} \quad \omega_{-k} = -\omega_k\,.$$

Moreover, one assumes the following time and space homogeneity properties. For any $k \in \mathbb{Z}^d\backslash\{0\}$, there exists a function $\sigma \mapsto A_k(\sigma)$ such that one has

$$\mathbb{E}[\underline{\Phi}^\varepsilon(t,k)\underline{\Phi}^\varepsilon(s,k')] = A_k\Big(\frac{t-s}{\varepsilon^2}\Big)\delta(k+k')\,, \tag{25}$$

with the following properties:

$$\sigma \mapsto A_k(\sigma) \text{ is even}, \qquad \forall|\sigma| > \tau \Longrightarrow A_k(\sigma) = 0\,, \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} |k|^3|A_k(\sigma)|d\sigma < C_1\,, \tag{26}$$

with C_1 being independent of ε.

In the right-hand side of (25), the term $A_k((t-s)/\varepsilon^2)$ stands for the time homogeneity assumption, while the term $\delta(k+k')$ represents the hypothesis of space homogeneity. Observe that the functions $\sigma \mapsto A_k(\sigma)$ can be extended by parity as functions defined on $\mathbb{R}$ and with their Fourier transforms given by

$$\hat{A}_k(s) = \int_{\mathbb{R}} A_k(\sigma)e^{-is\sigma}d\sigma\,. \tag{27}$$

As a consequence of the above hypotheses, one obtains, for ε small enough,

$$\begin{aligned}
\mathbb{D}^\varepsilon(t,v) &= \int_0^{\frac{t}{\varepsilon^2}} d\sigma\, \mathbb{E}[E^\varepsilon(t,x+\sigma v) \otimes E^\varepsilon(t-\varepsilon^2\sigma,x)] \\
&= \sum_{k\in\mathbb{Z}^d\backslash\{0\}} k\otimes k \int_0^{\frac{t}{\varepsilon^2}} A_k(\sigma)e^{-i(\omega_k-k\cdot v)\sigma}d\sigma
\end{aligned}$$

$$= \frac{1}{2} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} k \otimes k \,\fint\!\!\!\int_{\mathbb{R}} A_k(\sigma) e^{-i(\omega_k - k\cdot v)\sigma} d\sigma \,.$$

2.1 Properties of the Reynolds Electric Stress Tensor

Properties of the Reynolds electric stress tensor $\mathbb{D}^\varepsilon$ and of its limit as $\varepsilon \to 0$ are collected in:

Proposition 4 *Under assumptions* **H2** *and* **H3** *(see (23)–(25)):*

1. *The functions $s \mapsto \hat{A}_k(s)$ are real non-negative, analytic, and satisfy the estimate*

$$\sup_{s\in\mathbb{R}} \sum_{k\in\mathbb{Z}^d} |k|^3 |\hat{A}_k(s)| < C_1 \,. \tag{28}$$

2. *The limit of the Reynolds electric stress tensor*

$$\mathbb{D}^\varepsilon(t, v) = \int_0^{\frac{t}{\varepsilon^2}} d\sigma\, \mathbb{E}[E^\varepsilon(t, x + \sigma v) \otimes E^\varepsilon(t - \varepsilon^2\sigma, x)]$$

is a real non-negative symmetric diffusion matrix, which is analytic in the variable v and given by

$$\mathbb{D}(v) = \overline{\mathbb{D}^\varepsilon(t, v)} = \frac{1}{2} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} k \otimes k\, \hat{A}_k(\omega_k - k \cdot v)\,. \tag{29}$$

3. *For any $\psi,\ \phi \in \mathcal{D}(\mathbb{R}^d_v)$, one has*

$$\begin{aligned} &\overline{\int_{\mathbb{R}^+_t\times\mathbb{R}^d_v} (\nabla_v\phi(t, v))^T\, \mathbb{D}^\varepsilon(t, v)\nabla_v\psi(t, v) dv dt} = \\ &- \int_{\mathbb{R}^+_t\times\mathbb{R}^d_v} \phi(t, v)\nabla_v \cdot \big(\mathbb{D}(v)\nabla_v\psi(t, v)\big) dv dt\,. \end{aligned} \tag{30}$$

Proof The reality of $\hat{A}_k$ follows from the parity of the function $\sigma \mapsto A_k(\sigma)$. Then for any continuous and compactly supported function $\mathbb{R} \ni s \mapsto \phi(s)$, using (25) and obvious changes of variables in time, one obtains

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} A_k(t - s)\phi(s)\phi(t) ds dt = \\ &\varepsilon^4 \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\big[\phi(t/\varepsilon^2)\underline{\Phi}^\varepsilon(t, k)\big(\phi(s/\varepsilon^2)\underline{\Phi}^\varepsilon(s, k)\big)^\star\big] ds dt\,. \end{aligned} \tag{31}$$

Observe that the left-hand side of (31) is independent of ε, while the right-hand side is non-negative. As a consequence, one obtains

$$\int_{\mathbb{R}}\int_{\mathbb{R}} A_k(t-s)\phi(s)\phi(t)dsdt \geq 0\,,$$

and the positivity of $\hat{A}_k$ follows from the Bochner theorem (see [32]). Eventually, the fact that functions $\hat{A}_k(s)$ are analytic (an elementary version of the Paley–Wiener theorem) and satisfy estimate (28) is a direct consequence of (26). In the same way, the rest of the proof also follows directly from (24)–(25). □

2.2 Decorrelation

Assuming that the electric field is a stochastic function and introducing the expectation in the formula (19), one obtains

$$\begin{aligned}
&-\overline{\mathbb{E}\left[\fint dx \int_{\mathbb{R}^d_v} dv\, \phi(v)\nabla_v \cdot \left(\frac{E^\varepsilon f^\varepsilon}{\varepsilon}\right)\right]}\\
&=\overline{\mathbb{E}\left[\int_{\mathbb{R}^d_v} dv \fint dx \int_0^{\frac{t}{\varepsilon^2}} d\sigma f^\varepsilon(t-\sigma\varepsilon^2,x,v)\nabla_v\cdot\left(E^\varepsilon(t-\varepsilon^2\sigma,x)\otimes E^\varepsilon(t,x+\sigma v)\nabla_v\phi(v)\right)\right]}\,,
\end{aligned} \tag{32}$$

which leads to a “smooth” well-defined diffusion matrix but which also requires a decorrelation formula more or less of the following type:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^d_v} dv(\nabla_v\phi(v))^T \fint dx \int_0^{\frac{t}{\varepsilon^2}} d\sigma\; E^\varepsilon(t,x+\sigma v)\otimes E^\varepsilon(t-\varepsilon^2\sigma,x)\nabla_v f^\varepsilon(t-\sigma\varepsilon^2,x,v)\right]\\
&\simeq\\
&\int_{\mathbb{R}^d_v} dv\,(\nabla_v\phi(v))^T \fint dx \int_0^{\tau} d\sigma\, \mathbb{E}[E^\varepsilon(t,x+\sigma v)\otimes E^\varepsilon(t-\varepsilon^2\sigma,x)]\; \mathbb{E}[\nabla_v f^\varepsilon(t-\sigma\varepsilon^2,x,v)]\,,
\end{aligned}$$

and this is the object of the following lemma and proposition.

Lemma 1 (Time Decorrelation Property Between f^ε and E^ε) *Assume* **H2** *or (23). Suppose that the random initial data f_0^ε and the electric field E^ε are independent. Then the operator $E^\varepsilon(s)\cdot\nabla_v$ is independent of $f^\varepsilon(t)$ as soon as $s \geq t+\varepsilon^2\tau$.*

Proof From the Duhamel formula

$$f^\varepsilon(t) = S_t^\varepsilon f_0^\varepsilon - \frac{1}{\varepsilon}\int_0^t d\sigma\; S_{t-\sigma}^\varepsilon E^\varepsilon(\sigma)\nabla_v f^\varepsilon(\sigma)\,, \tag{33}$$

we observe that $f^\varepsilon(t)$ depends only on f_0^ε and $E^\varepsilon(\sigma)$ for $\sigma \leq t$. Since f_0^ε is independent of $E^\varepsilon(t)$, $\forall t \in \mathbb{R}$, and since the electric fields $E^\varepsilon(s)$ and $E^\varepsilon(t)$ are independent as soon as $s > t + \varepsilon^2\tau$ (assumption **H2** or (23)), Lemma 1 follows directly from (33). □

In the Duhamel formula connecting the solution from the time $t - \varepsilon^2\tau$ to the time t,

$$f^\varepsilon(t) = S^\varepsilon_{\varepsilon^2\tau} f^\varepsilon(t-\varepsilon^2\tau) - \frac{1}{\varepsilon}\int_0^{\varepsilon^2\tau} d\sigma\, S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v f^\varepsilon(t-\sigma)\,, \tag{34}$$

we insert for $f^\varepsilon(t-\sigma)$ the Duhamel formula connecting the solution from the time $t - 2\varepsilon^2\tau$ to the time $t-\sigma$ to obtain

$$\begin{aligned} f^\varepsilon(t) &= S^\varepsilon_{\varepsilon^2\tau} f^\varepsilon(t-\varepsilon^2\tau) - \frac{1}{\varepsilon}\int_0^{\varepsilon^2\tau} d\sigma\, S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v (S^\varepsilon_{-\sigma} S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t-2\varepsilon^2\tau)) \\ &+\frac{1}{\varepsilon^2}\int_0^{\varepsilon^2\tau} d\sigma \int_0^{2\varepsilon^2\tau-\sigma} ds\, S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v(S^\varepsilon_s(E^\varepsilon(t-\sigma-s)\cdot\nabla_v f^\varepsilon(t-\sigma-s)))\,, \end{aligned} \tag{35}$$

which provides the essential tool for the needed decorrelation property according to the following:

Proposition 5 *Assume that the vector field E^ε satisfies Hypotheses* **H1** *or (22) and* **H2** *or (23); then for the expectation of the Fick term, one obtains*

$$\begin{aligned} &-\nabla_v\cdot\mathbb{E}\left[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right] \\ &= \frac{1}{\varepsilon^2}\int_0^{\varepsilon^2\tau} d\sigma \fint dx\, \mathbb{E}[E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v S^\varepsilon_{-\sigma}]\mathbb{E}[f^\varepsilon(t-2\varepsilon^2\tau)] + \mathbb{E}[\mu^\varepsilon_t]\,, \end{aligned}$$

with

$$\mu^\varepsilon_t = -\frac{1}{\varepsilon^3}\int_0^{\varepsilon^2\tau} d\sigma \int_0^{2\varepsilon^2\tau-\sigma} ds \fint dx\, E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v S^\varepsilon_s E^\varepsilon(t-\sigma-s)\cdot\nabla_v f^\varepsilon(t-\sigma-s)]\,.$$

Proof Applying operator $E^\varepsilon(t)\cdot\nabla_v$ to (35), and then applying successively the average in space and the expectation value, we obtain

$$-\nabla_v\cdot\mathbb{E}\left[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right] = \frac{1}{\varepsilon}\fint dx\, \mathbb{E}\big[E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_{\varepsilon^2\tau} f^\varepsilon(t-\varepsilon^2\tau)\big]$$

$$+\frac{1}{\varepsilon^2}\int_0^{\varepsilon^2\tau} d\sigma \fint dx\, \mathbb{E}\Big[E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v S^\varepsilon_{-\sigma} S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t-2\varepsilon^2\tau)\Big]+\mathbb{E}[\mu^\varepsilon_t]\,, \tag{36}$$

with

$$\mu^\varepsilon_t = -\frac{1}{\varepsilon^3}\int_0^{\varepsilon^2\tau} d\sigma \int_0^{2\varepsilon^2\tau-\sigma} ds \fint dx\, E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v S^\varepsilon_s E^\varepsilon(t-\sigma-s)\cdot\nabla_v f^\varepsilon(t-\sigma-s)\,.$$

Using Lemma 1, we obtain that $f^\varepsilon(t)$ is independent of $E^\varepsilon(s)\cdot\nabla_v$ as soon as $s \geq t+\varepsilon^2\tau$. Then, using hypothesis **H1**, we obtain

$$\mathbb{E}\big[E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_{\varepsilon^2\tau} f^\varepsilon(t-\varepsilon^2\tau)\big] = \mathbb{E}\big[E^\varepsilon(t)\cdot\nabla_v\big] S^\varepsilon_{\varepsilon^2\tau}\mathbb{E}\big[f^\varepsilon(t-\varepsilon^2\tau)\big] = 0\,,$$

and the first term of the right-hand side of (36) vanishes. Since Proposition 1 implies that $E^\varepsilon(t)\cdot\nabla_v$ and $E^\varepsilon(t-\sigma)\cdot\nabla_v$ are independent of $f^\varepsilon(t-2\varepsilon^2\tau)$, for $0 \leq \sigma \leq \varepsilon^2\tau$, we obtain from (36)

$$\frac{1}{\varepsilon^2}\int_0^{\varepsilon^2\tau} d\sigma \fint dx\, \mathbb{E}\Big[E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v S^\varepsilon_{-\sigma} S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t-2\varepsilon^2\tau)\Big]$$
$$= \frac{1}{\varepsilon^2}\int_0^{\varepsilon^2\tau} d\sigma \fint dx\, \mathbb{E}\big[E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v S^\varepsilon_{-\sigma}\big]\,\mathbb{E}\Big[S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t-2\varepsilon^2\tau)\Big]\,.$$

2.3 Weak Limits

The asymptotic behavior of the error term μ^ε_t as $\varepsilon \to 0$ is given by:

Proposition 6 *For any $\phi \in \mathcal{D}(\mathbb{R}^+_t \times \mathbb{R}^d_v)$, one obtains*

$$|\langle \mu^\varepsilon_t\,,\,\phi\rangle| \leq \varepsilon\tau^4 C(\phi)\mathbb{E}\Big[\|E^\varepsilon\|^3_{L^\infty(\mathbb{R}^+_t;W^{2,\infty}(\mathbb{T}^d))}\Big]\,.$$

Proof First changing (σ, s) into $(\varepsilon^2\sigma, \varepsilon^2 s)$, with any $\phi \in \mathcal{D}(\mathbb{R}^+_t \times \mathbb{R}^d_v)$, one obtains

$$\langle \mu^\varepsilon_t\,,\,\phi\rangle = -\int_{\mathbb{R}^+_t\times\mathbb{R}^d_v} dt\,dv\,\phi(t,v)$$
$$\frac{1}{\varepsilon^3}\int_0^{\varepsilon^2\tau} d\sigma \int_0^{2\varepsilon^2\tau-\sigma} ds \fint dx\, E^\varepsilon(t)\cdot\nabla_v S^\varepsilon_\sigma E^\varepsilon(t-\sigma)\cdot\nabla_v S^\varepsilon_s E^\varepsilon(t-\sigma-s)\cdot\nabla_v f^\varepsilon(t-\sigma-s)$$

$$= -\varepsilon \iint_{\mathbb{R}_t^+ \times \mathbb{R}_v^d} dt dv\, \phi(t,v) \iint_0^\tau d\sigma \iint_0^{2\tau-\sigma} ds \fint dx$$

$$E^\varepsilon(t) \cdot \nabla_v S^\varepsilon_{\varepsilon^2\sigma} E^\varepsilon(t-\varepsilon^2\sigma) \cdot \nabla_v S^\varepsilon_{\varepsilon^2 s} E^\varepsilon(t-\varepsilon^2(\sigma+s)) \cdot \nabla_v f^\varepsilon(t-\varepsilon^2(\sigma+s)) .$$

Then, with several integrations by part and using the fact that ${S^\varepsilon}_t^* = S^\varepsilon_{-t}$, one obtains (see [1])

$$\langle \mu_t^\varepsilon\,,\, \phi\rangle = -\varepsilon \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_0^\tau d\sigma \int_0^{2\tau-\sigma} ds \fint dx$$

$$f^\varepsilon(t-\varepsilon^2(\sigma+s)) E^\varepsilon(t-\varepsilon^2(\sigma+s)) \cdot \nabla_v (S^\varepsilon_{-\varepsilon^2 s} E^\varepsilon(t-\varepsilon^2\sigma) \cdot \nabla_v (S^\varepsilon_{-\varepsilon^2\sigma} E^\varepsilon(t) \cdot \nabla_v \phi)) . \tag{37}$$

In the last line of (37) appears the term

$$E^\varepsilon(t-\varepsilon^2(\sigma+s)) \cdot \nabla_v (S^\varepsilon_{-\varepsilon^2 s} E^\varepsilon(t-\varepsilon^2\sigma) \cdot \nabla_v (S^\varepsilon_{-\varepsilon^2\sigma} E^\varepsilon(t) \cdot \nabla_v \phi)) ,$$

which contains at most second-order derivatives with respect to v of expressions of the form $E^\varepsilon(s, x+\tilde\sigma v)$. With τ finite, $x \in \mathbb{T}^d$, and with the introduction of a test function $\phi \in \mathcal{D}(\mathbb{R}_t^+ \times \mathbb{R}_v^d)$, the support of the integrand is bounded in $\mathbb{R}_t^+ \times \mathbb{T}^d \times \mathbb{R}_v^d$. Then, with a crude estimate (that could be improved), one obtains

$$|\langle \mu_t^\varepsilon\,,\, \phi\rangle| \le \varepsilon \tau^4 C(\phi) \| E^\varepsilon \|^3_{L^\infty(\mathbb{R}_t^+; W^{2,\infty}(\mathbb{T}^d))} . \tag{38}$$

Finally, taking the expectation of (38), one concludes the proof of Lemma 6. □

The diffusion limit is given by:

Proposition 7 *For any $\phi \in \mathcal{D}(\mathbb{R}_t^+ \times \mathbb{R}_v^d)$, one obtains*

$$\overline{\int_{\mathbb{R}_t^+ \times \mathbb{R}_v^d} dt dv\, \phi \nabla_v \cdot \mathbb{E}\left[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right]} = -\int_{\mathbb{R}_t^+ \times \mathbb{R}_v^d} dt dv\, \overline{f^\varepsilon}(t,v) \nabla_v \cdot \big(\mathbb{D}(v) \nabla \phi(t,v)\big) . \tag{39}$$

Proof Knowing already from Proposition 6 that $\mu_t^\varepsilon \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}_t^+ \times \mathbb{R}_v^d)$ as $\varepsilon \to 0$, one obtains

$$\int_{\mathbb{R}_t^+ \times \mathbb{R}_v^d} dt dv\, \phi \nabla_v \cdot \mathbb{E}\left[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right] = I^\varepsilon , \tag{40}$$

with

$$I^\varepsilon := -\int_{\mathbb{R}_t^+} dt \int_{\mathbb{R}_v^d} dv \frac{1}{\varepsilon^2} \int_0^{\varepsilon^2 \tau} d\sigma$$
$$⨍ dx\, \phi(t,v) \mathbb{E}\left[E^\varepsilon(t) \cdot \nabla_v S_\sigma^\varepsilon E^\varepsilon(t-\sigma) \cdot \nabla_v S_{-\sigma}^\varepsilon\right] \mathbb{E}\left[S_{2\varepsilon^2\tau}^\varepsilon f^\varepsilon(t-2\varepsilon^2\tau)\right].$$

After expanding the integrand of I^ε, using an integration by parts in v, and changing σ into $\varepsilon^2\sigma$, one obtains

$$I^\varepsilon := \int_{\mathbb{R}_t^+} dt \int_{\mathbb{R}_v^d} dv \int_0^\tau d\sigma ⨍ dx\, \nabla_v \phi^T\, \mathbb{E}[E^\varepsilon(t,x) \otimes E^\varepsilon(t-\varepsilon^2\sigma, x-v\sigma)]$$
$$\mathbb{E}[(\sigma-2\tau)(\nabla_x f)(t-2\varepsilon^2\tau, x-2v\tau, v) + (\nabla_v f)(t-2\varepsilon^2\tau, x-2v\tau, v)].$$

Using the change of variables $(t,\ x,\ v) \to (t' = t - 2\varepsilon^2\tau,\ x' = x - 2v\tau,\ v' = v)$ and integration by parts in (x, v), one obtains

$$I^\varepsilon := -\frac{1}{(2\pi)^d} \int_{\mathbb{R}_t} dt \int_{\mathbb{R}_v^d} dv \int_0^\tau d\sigma \int_{\mathbb{R}_x^d} dx\, \mathbb{E}[f^\varepsilon(t,x,v)] \Psi^\varepsilon(t,\sigma,x,v),$$

with

$$\Psi^\varepsilon(t,\sigma,x,v) = \mathbb{1}_{[-2\varepsilon^2\tau, +\infty[}(t) \mathbb{1}_{\{\mathbb{T}^d - 2v\tau\}}(x)\ (\nabla_v \cdot + (\sigma-2\tau)\nabla_x \cdot)$$
$$\mathbb{E}[E^\varepsilon(t-\varepsilon^2(\sigma-2\tau), x-v(\sigma-2\tau)) \otimes E^\varepsilon(t+2\varepsilon^2\tau, x+2v\tau)] \nabla_v \phi(t+2\varepsilon^2\tau, v).$$

The domain of integration of the integral I^ε is a compact set K of $\mathbb{R}_t \times \mathbb{R}_v^d \times [0,\tau]_\sigma \times \mathbb{T}^d$. We already know that $\mathbb{E}[f^\varepsilon(t,x,v)] \rightharpoonup \overline{f^\varepsilon}(t,v)$ in $L^\infty(K)$ weak$-\star$. It remains to show the strong convergence in $L^1(K)$ of Ψ^ε to a suitable cluster point $\overline{\Psi^\varepsilon}$. For this, using (24), one obtains

$$\Psi^\varepsilon(t,\sigma,x,v) = \mathbb{1}_{[-2\varepsilon^2\tau, +\infty[}(t) \mathbb{1}_{\{\mathbb{T}^d - 2v\tau\}}(x)\ (\nabla_v \cdot + (\sigma-2\tau)\nabla_x \cdot)$$
$$\sum_{k,k' \in \mathbb{Z}^d} -k \otimes k'\, e^{i(k+k')\cdot x} e^{i2(k+k')\cdot v\tau} e^{-i2(\omega_k+\omega_{k'})\tau} e^{-i(\omega_k+\omega_{k'})\frac{t}{\varepsilon^2}} e^{-i(\omega_k - k\cdot v)\sigma}$$
$$\mathbb{E}[\underline{\Phi}^\varepsilon(t-\varepsilon^2(\sigma-2\tau),k) \underline{\Phi}^\varepsilon(t+2\varepsilon^2\tau,k')] \nabla_v \phi(t+2\varepsilon^2\tau, v).$$

Without space homogeneity, we observe that the term $\exp\bigl(-i(\omega_k+\omega_{k'})t/\varepsilon^2\bigr)$ does not converge pointwise almost everywhere in time, which prevents strong convergence in $L^1(K)$ of the function Ψ^ε. By constrast, using the spatio-temporal homogeneity property (25), one obtains

$$\Psi^\varepsilon(t,\sigma,x,v) = \mathbb{1}_{[-2\varepsilon^2\tau,\,+\infty[}(t)\mathbb{1}_{\{\mathbb{T}^d-2v\tau\}}(x)\ (\nabla_v\cdot+(\sigma-2\tau)\nabla_x\cdot) \sum_{k\in\mathbb{Z}^d} A_k(\sigma)e^{-i(\omega_k-k\cdot v)\sigma}k\otimes k\nabla_v\phi(t+2\varepsilon^2\tau,v)\,.$$

Using regularity properties (26) and Lebesgue dominated convergence theorem, one obtains that Ψ^ε converges in $L^1(K)$ strongly toward the cluster point $\overline{\Psi^\varepsilon}$, which is defined by

$$\overline{\Psi^\varepsilon(t,\sigma,x,v)} = \mathbb{1}_{\mathbb{R}^+}(t)\mathbb{1}_{\{\mathbb{T}^d-2v\tau\}}(x)\ (\nabla_v\cdot+(\sigma-2\tau)\nabla_x\cdot) \sum_{k\in\mathbb{Z}^d} A_k(\sigma)e^{-i(\omega_k-k\cdot v)\sigma}k\otimes k\nabla_v\phi(t,v)\,.$$

Using properties (26) for A_k, and passing to the limit $\varepsilon\to 0$ in I^ε, one obtains

$$\overline{I^\varepsilon} = -\int_{\mathbb{R}_t^+} dt\int_{\mathbb{R}_v^d} dv\,\overline{f^\varepsilon}(t,v)\nabla_v\cdot\Big(\frac{1}{2}\sum_{k\in\mathbb{Z}^d} k\otimes k\int_{\mathbb{R}} d\sigma\, A_k(\sigma)e^{-i(\omega_k-k\cdot v)\sigma}\nabla_v\phi(t,v)\Big)\,.$$

Using this last equation, definitions (27) and (29), and passing to the limit $\varepsilon\to 0$ in (40), ones obtains (39), which ends the proof of Proposition 7. □

2.4 The Basic Stochastic Theorem

From the above derivation, one deduces:

Theorem 1 *Let* $\{E^\varepsilon(t,x;\omega)\}_{\omega\in\Omega} = \{-\nabla\Phi^\varepsilon(t,x;\omega)\}_{\omega\in\Omega}$ *be a family of stochastic (with respect to the random variable* $\omega\in\Omega$*) gradient vector fields. Assume that such vector fields satisfy the* ε*-independent local-in-time regularity hypothesis*

$$\forall\varepsilon>0 \quad and \quad \forall T>0,\quad \sup_{0<t<T}\|E^\varepsilon(t)\|_{W^{2,\infty}(\mathbb{T}^d)} \le C(T)\,,$$

and the following detailed ergodicity hypotheses. With

$$\forall k\in\mathbb{Z}^d\backslash\{0\},\quad E^\varepsilon(t,k) = \fint E^\varepsilon(t,x)e^{-ik\cdot x}dx = \underline{E}^\varepsilon(t,k)e^{-i\frac{\omega_k t}{\varepsilon^2}} = -ik\underline{\Phi}^\varepsilon(t,k)e^{-i\frac{\omega_k t}{\varepsilon^2}}\,,$$

there exist a constant $\tau\in(0,+\infty)$ *and functions* $\sigma\mapsto A_k(\sigma)$, $k\in\mathbb{Z}^d\backslash\{0\}$, *such that*

$$\mathbb{E}[\underline{\Phi}^\varepsilon(t,k)\underline{\Phi}^\varepsilon(s,k')] = A_k\Big(\frac{t-s}{\varepsilon^2}\Big)\delta(k+k'),$$

$$\sigma \mapsto A_k(\sigma) \text{ is even,} \quad |\sigma| > \tau \Longrightarrow A_k(\sigma) = 0, \quad \text{and} \quad \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} |k|^3|A_k(\sigma)|d\sigma < C_1,$$

with C_1 being independent of ε.

Then:

1. *For all $k \in \mathbb{Z}^d\backslash\{0\}$, the Fourier transform of the function $\sigma \mapsto A_k(\sigma)$ is non-negative, and the bounded diffusion matrix*

$$\mathbb{D}(v) = \frac{1}{2}\sum_{k\in\mathbb{Z}^d\backslash\{0\}} k\otimes k\ \hat{A}_k(\omega_k - k\cdot v) \tag{41}$$

is symmetric non-negative and analytic in the variable v.

2. *Define by $V \subset L^2(\mathbb{R}^d_v)$ the closure of the space of functions $\phi \in \mathcal{D}(\mathbb{R}^d_v)$ for the norm*

$$\|\phi\|_V^2 = \|\phi\|^2_{L^2(\mathbb{R}^d_v)} + \int_{\mathbb{R}^d_v}\big(\mathbb{D}(v)\nabla_v\phi\,,\ \nabla_v\phi\big).$$

Then for any $f_0(v) \in L^2(\mathbb{R}^d_v)$, there exists a unique solution of the following problem: find

$$f(t,v) \in C\big(\mathbb{R}^+_t;\ L^2(\mathbb{R}^d_v)\big) \cap L^2\big(\mathbb{R}^+_t;\ V\big) \quad \text{with} \quad f(0,v) = f_0(v), \tag{42}$$

such that f is the solution (in the sense of $\mathcal{D}'(\mathbb{R}^+_t\times\mathbb{R}^d_v)$) of the diffusion equation

$$\partial_t f - \nabla_v\cdot(\mathbb{D}(v)\nabla_v f) = 0. \tag{43}$$

3. *Since $v \mapsto \mathbb{D}(v)$ is regular (analytic), the time derivative of the solution of (43) is well-defined (say in $L^\infty(\mathbb{R}^+_t;\ H^{-2}(\mathbb{R}^d_v))$); hence, the initial condition $f(0,v) = f_0(v)$ is well-defined, and with such initial data, this solution coincides with the unique solution of the problem (42)–(43).*
4. *Any cluster point $\overline{f^\varepsilon}$, in the $L^\infty_{\text{loc}}(\mathbb{R}^+_t\times\mathbb{R}^d_v)$ weak$-\star$ topology, of the family $\{\mathbb{E}[f^\varepsilon]\}_\varepsilon$ with f^ε solution of the stochastic Liouville equation*

$$\varepsilon^2\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \varepsilon E^\varepsilon\cdot\nabla_v f^\varepsilon = 0, \qquad f^\varepsilon(0,x,v) = f_0(x,v),$$

is a function $\overline{f^\varepsilon}(t,v)$ independent of x and solution of (43) with initial data given by

$$\overline{f^\varepsilon}(0,x) = \fint f_0(x,v)dx.$$

5. *Eventually by a uniqueness argument, it is not a subsequence of* $\mathbb{E}[f^\varepsilon]$ *but the whole family that converges to the solution of the diffusion equation.*

Proof The point 1 follows directly from the points 1 and 2 of the Proposition 4. The point 2 is a classical result of variational theory (see [25]). The purpose of the point 3 is to prove the regularity (42) for solutions in the sense of distribution. This is easily done by considering standard regularizations of f in velocity such that $f_\epsilon = \varrho_\epsilon *_v f$ (with ϱ_ϵ a Freidrichs mollifier). For the point 4, the limit equation results from relation (39) of Proposition 7. Then, to prove the time continuity in $H^{-2}(\mathbb{R}^d_v)$, one considers the equation

$$\partial_t \mathbb{E}\left[\fint dx\, f^\varepsilon(t)\right] + \nabla_v \cdot \mathbb{E}\left[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right] = 0\,,$$

and from estimate (38) and Eq. (40), one deduces that

$$\partial_t \mathbb{E}\left[\fint dx\, f^\varepsilon(t)\right]$$

is uniformly bounded in $L^\infty(\mathbb{R}^+_t; H^{-2}(\mathbb{R}^d_v))$, and this gives a uniform estimate on the time continuity, which is enough to complete the proof of the point 5. □

3 Returning to the Vlasov–Poisson Equations

3.1 Classical Stability Results for Nonlinear and Linearized Vlasov–Poisson Equations

As above, solutions of the Vlasov–Poisson equations are considered on $\mathbb{R}^+_t \times \mathbb{T}^d \times \mathbb{R}^d_v$ and, before any rescaling, involve a probability density f, which is solution of the Liouville equation

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0\,.$$

The electric field E derives from a self-consistent potential given in terms of the density ρ by

$$-\Delta\Phi(t,x) = \rho(t,x) = \int_{\mathbb{R}^d_v} f(t,x,v)dv - 1\,, \qquad E(t,x) = -\nabla\Phi(t,x)\,.$$

Existence, uniqueness, and persistence of regularity (with regular initial data) are classical (see [16]) and in particular for propagation of analyticity, see [2]). However, these regularity properties may not be uniform with respect to time and scaling parameters. Moreover, Fourier transform with respect to the velocity and

Laplace transform with respect to time are used. They are denoted as follows:

$$\begin{aligned}
\mathcal{L}h(\lambda, v) &= \int_0^{\infty} e^{-\lambda s} h(s, v) ds\,, \\
h(k, v) &= \fint h(x, v) e^{-ik\cdot x} dx\,, \\
\mathcal{F}_v G(\xi) &= \int_{\mathbb{R}^d} G(v) e^{-iv\cdot\xi} dv\,.
\end{aligned} \tag{44}$$

Properties that are independent of time are recalled below. With $f(0, x, v) \in L^p(\mathbb{T}^d \times \mathbb{R}^d_v)$, $1 \le p \le \infty$, being a positive density of mass 1 , one has

$$\begin{aligned}
&\forall t \in \mathbb{R}^+\,, \quad f(t, x, v) \ge 0\,, \\
&1 \le p < \infty\,, \quad \int_{\mathbb{T}^d\times\mathbb{R}^d_v} |f(t, x, v)|^p dv dx = \int_{\mathbb{T}^d\times\mathbb{R}^d_v} |f(0, x, v)|^p dv dx, \\
&\forall t \in \mathbb{R}^+\,, \quad \frac{d}{dt}\left(\fint \int_{\mathbb{R}^d_v} \frac{|v|^2}{2} f(t, x, v) dv dx + \frac{1}{2}\fint |E(t, x)|^2 dx\right) = 0\,.
\end{aligned}$$

With the above estimates, a classical interpolation inequality for the density ρ (see [4, 9, 10]) and standard elliptic regularity estimates, one also deduces the following time uniform regularity:

$$\text{In any dimension } d\,, \qquad \|E\|_{L^\infty(\mathbb{R}^+_t; W^{1,1+2/d}(\mathbb{T}^d))} \le c_0 < \infty.$$

3.2 Spectral Properties of Linearized Vlasov–Poisson Equations

Below, to use in short-time quasilinear approximation, some very classical facts are recorded. Any density $v \mapsto G(v) \ge 0$ with $\int_{\mathbb{R}^d_v} G(v) dv = 1$ is a stationary solution of the Vlasov–Poisson equations (with $E(t, x) \equiv 0$). Hence, for $f(t, x, v) = G(v) + \varepsilon h(t, x, v)$, the ε first-order term is the solution of the following linearized equation:

$$\begin{aligned}
&\partial_t h + v \cdot \nabla_x h + E[h] \cdot \nabla_v G = 0\,, \\
&\fint dx \int_{\mathbb{R}^d_v} h(t, x, v) dv = 0\,, \quad E[h](t, x) = \nabla\Delta^{-1} \int_{\mathbb{R}^d_v} h(t, x, v) dv\,.
\end{aligned} \tag{45}$$

By standard perturbation (see [19]), the operator $\mathcal{T}_G : h \to \mathcal{T}_G h = -(v \cdot \nabla_x h + E[h] \cdot \nabla_v G)$ is the generator of a strongly continuous group $e^{t\mathcal{T}_G}$ (particularly in $L^2(\mathbb{T}^d \times \mathbb{R}^d_v)$). Next one observes that in this space, the map $h \mapsto E[h] \cdot \nabla_v G$ is a compact operator. Hence, with the Duhamel formula,

$$e^{t\mathcal{T}_G}h_0 = S_t h(0) - \int_0^t d\sigma\, S_{t-\sigma} E[h](\sigma)\cdot\nabla_v G(v)\,,$$

one deduces that for any $\alpha > 1$, the spectra of $e^{t\mathcal{T}_G}$, contained in the region $\{\mu \in \mathbb{C} \,|\, |\mu| \geq \alpha > 1\}$, are a finite sum of eigenvalues with finite multiplicity, which are the images of the poles of the resolvent of the generator $-\mathcal{T}_G$, i.e., complex numbers $\lambda_m\,,\ \ \Re\lambda_m > 0$, such that the equation

$$\lambda_m h + v\cdot\nabla_x h + E[h]\cdot\nabla_v G = 0\,,\quad h\in L^2(\mathbb{T}^d\times\mathbb{R}^d_v)\,,$$

has a non-trivial solution. Using Fourier series on $\mathbb{T}^d$, this means that there exists at least one $h(k_m, v) \in L^2(\mathbb{R}^d_v)$ (may be more than one if the multiplicity of λ_m is greater than one) such that, taking into account the relation between the Fourier components of the electric field and of the density, namely

$$E[h](k_m) = -\frac{ik_m}{|k_m|^2}\rho[h](k_m)\,,$$

one has the "dispersion equation"

$$1 - \int_{\mathbb{R}^d_v}\frac{ik_m}{|k_m|^2}\cdot\frac{\nabla_v G(v)}{\lambda_m + ik_m\cdot v}dv = 0\,. \tag{46}$$

Since $G(v)$ is real, one observes that if (λ_m, k_m) is a solution, then the same is true for $((\lambda_m)^\star, -k_m)$, and that in this case one obtains for the Fourier component of h

$$h_{\lambda_m}(k_m, v) = -\frac{1}{(\lambda_m + ik_m\cdot v)}E_{\lambda_m}(k_m)\cdot\nabla_v G(v)\,.$$

As a consequence (assuming for sake of simplicity that $\lambda_m = \gamma_m + i\omega_m$ is a simple root of the analytic equation (46)), one observes that $h_m(t,x,v)$, the solution of Eq. (45), and the electric field

$$E_m(t,x) = e^{ik_m x + \lambda_m t}E_{\lambda_m}(k_m)$$

are bound by the relation, for any time t,

$$h_m(t,x,v) = -\frac{e^{ik_m x+\lambda_m t}}{(\lambda_m + ik_m\cdot v)}E_{\lambda_m}(k_m)\cdot\nabla_v G\,.$$

On the other hand, for any λ_m, introduce the Kato projector (see [19] page 178), on the eigenspace corresponding to the eigenvalue λ_m, defined by

$$P_m h_0(x,v) = \frac{1}{2i\pi}\int_{\Gamma_m} (\lambda I - \mathcal{T}_G)^{-1} h_0 d\lambda\,,$$

where Γ_m is a "small" oriented contour of the complex half- plane $\Re\lambda > 0$ containing only λ_m in its interior. Since for any $\delta > 0$ there is a finite number of eigenvalues in the region

$$\left\{\lambda \in \mathbb{C} \mid 0 < \delta \le \Re\lambda \le \Lambda := \sup \Re\lambda_m \right\},$$

one obtains (assuming that these eigenvalues are simple), for any real density $h(t,x,v)$ solution of the linearized equation (45), the following asymptotic expansion:

$$h(t,x,v) = \sum_{\{\lambda_m\}_m \mid 0<\delta\le\gamma_m\le\Lambda} e^{(\gamma_m + i\omega_m)t} e^{ik_m\cdot x} P_m h_0 + O(e^{\delta t})\,. \tag{47}$$

Using an explicit Laplace transform (which is the forerunner of the Dunford–Kato calculus used above), one tries to move the integration contour to the left half-plane, i.e., on the interval $\lambda + i\sigma$ with $\Re\lambda < 0$ and $-\infty < \sigma < +\infty$. In doing so, one generates a singularity at the point $\omega = -k\cdot v$ and by the Plemelj formula a term of the form $i\pi f(-ik\cdot v)$ that makes sense only under the hypothesis that the function f is analytic (see [21] chapter 8). These are standard methods in linear scattering theory (see [19, 24]) that in the absence of more specific information are based on analyticity hypotheses and convergence in ultra-distributions (dual of analytic functions). These tools were introduced by Case and Zweifel [6, 7], extended by Sebastião e Silva [29], and later systematized by Degond [8].

3.3 Remark About the Landau Damping, Comparison with the Behavior of the Vector Field in the Rescaled Equation

In 1946, Landau [22] observed that in the absence of unstable modes (no solutions λ of Eq. (46) with $\Re\lambda > 0$) the electric field goes exponentially fast to zero as $t \to \infty$. Following the recent version of Grenier et al. [17], we assume that the profile $v \mapsto G(v)$ and the initial data $(x,v) \mapsto h_0(x,v)$ are analytic functions. Using notation (44), Laplace transform with respect to time, Fourier transform with respect to x, and Fourier transform with respect to v are used. They are denoted as in Sect. 3.1. We first focus on the behavior of the density $\rho[h]$, of the solution h of the linearized equation (45), given by

$$\rho[h](t,x) = \int_{\mathbb{R}^d_v} h(t,x,v)dv\,.$$

Using Fourier–Laplace transformations, we obtain, for $\Re\lambda > 0$, the relation

$$(1 + K_G(\lambda, k))\mathcal{L}\rho[h](\lambda, k) = \int_{\mathbb{R}^d_v} \frac{h_0(k, v)}{\lambda + ik \cdot v} dv \quad \text{with} \quad K_G(\lambda, k) = -\int_{\mathbb{R}^d_v} \frac{ik}{|k|^2} \cdot \frac{\nabla_v G(v)}{\lambda + ik \cdot v} dv \,.$$

Then, following [27], one observes that

$$K_G(\lambda, k) = \int_0^\infty e^{-\lambda s} s(\mathcal{F}_v G)(ks) ds \quad \text{and} \quad \int_{\mathbb{R}^d_v} \frac{\hat{h}_0(k, v)}{\lambda + ik \cdot v} dv = \int_0^\infty e^{-\lambda s} (\mathcal{F}_v h_0)(k, ks) ds \,.$$

Therefore, as in [27], one proposes a stronger criterion for the absence of solution λ_m with $\Re\lambda_m > 0$ for Eq. (46), which is

$$\exists \kappa_0 > 0, \text{ such that} \quad \inf_{k \in \mathbb{Z}^d, \Re\lambda > 0} \left| 1 + \int_0^\infty e^{-\lambda s} s \mathcal{F}_v G(ks) ds \right| \geq \kappa_0 > 0 \,. \tag{48}$$

Then, we obtain for $\Re\lambda > 0$ the solution

$$\mathcal{L}\rho[h](\lambda, k) = \frac{1}{(1 + K_G(\lambda, k))} \int_0^\infty e^{-s\lambda} (\mathcal{F}_v) h_0(k, ks) ds \,.$$

With the hypothesis of analyticity, by the Paley–Wiener Theorem, there exist C and θ_0 such that one has

$$|\mathcal{F}_v G(ks)| \leq C e^{-\theta_0 |k| s} \,, \qquad \text{and} \qquad |\mathcal{F}_v h_0(k, ks)| \leq C e^{-\theta_0 |k|} \,.$$

As a consequence, using (48) for any $k \in \mathbb{Z} \backslash \{0\}$, the functions

$$K_G(\lambda, k) = \int_0^\infty e^{-\lambda s} s \mathcal{F}_v G(ks) ds \quad \text{and} \quad S(\lambda, k) = \int_0^\infty e^{-s\lambda} \mathcal{F}_v h_0(k, ks) ds$$

can be extended (for the density $\rho[h]$, the integration with respect to v leads to an extension behind the imaginary axis without extra singularity) as analytic functions in the region $\Re\lambda > -\theta_0 |k|$. Eventually, one obtains (see [17]) that there exists $\theta_1 > 0$ such that

$$\text{for} \quad \Re\lambda \geq -\theta_1 |k|, \quad |\mathcal{L}\rho[h](\lambda, k)| \leq \frac{C_1}{1 + |k|^2 + |\Im\lambda|^2} \,. \tag{49}$$

This gives the exponential decay for $\rho[h]$ and $E[h]$ when G is analytic. This is sufficient for the present discussion (extension to an initial data f_0 belonging only to a Gevrey space with Gevrey index $\gamma \in (\frac{1}{3}, 1]$ uses the presence of the term $|k|^2$ in (49)). Of course, the nonlinearity requires more sophisticated analysis in particular in the interaction of modes, which is the classical problem of the echoes. Details can be found in [17], where the following theorem is obtained.

Theorem 2 *Assume for the initial data,*

$$f_0(x, v) = G(v) + \varepsilon h_0(x, v), \tag{50}$$

that $G(v)$ and $h_0(x, v)$ are analytic, while the basic profile $G(v)$ satisfies the stability estimate given by the formula (48). Then, for ε small enough, the corresponding solution exhibits the Landau damping effect, i.e., as $t \to \infty$, the electric field $E(t, x)$ goes exponentially fast to zero.

As observed in the introduction, the ergodicity of the torus $\mathbb{T}^d$ implies that for any $0 < T < \infty$ the solution $f^\varepsilon(t, x, v)$, of the rescaled equation, converges in $L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d_v)$ weakly$-\star$ to an x-independent function. Hence, with the Poisson equation, the electric field converges also in $L^\infty(0, T; L^2(\mathbb{T}^d))$ weakly$-\star$ to zero. Therefore, this property would justify the term "baby Landau damping." In the above situation, genuine Landau damping would correspond to strong convergence (for any $\delta > 0$ and $\delta < T < \infty$) in $L^\infty(\delta, T; L^2(\mathbb{T}^d))$. Strong convergence will be the counterpart, in the present situation, of Theorem 2. In fact, one has:

Theorem 3 *For solutions of the rescaled Vlasov–Poisson equations,*

$$\begin{aligned} &\varepsilon^2 \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \varepsilon E^\varepsilon \cdot \nabla_v f^\varepsilon = 0\,, \\ &\nabla_x \cdot E^\varepsilon = \rho(t, x) = \int_{\mathbb{R}^d_v} f(t, x, v) dv - 1\,, \end{aligned} \tag{51}$$

near an equilibrium

$$f_0(x, v) = G(v) + h_0(x, v)\,,$$

with analytic data and profile satisfying the stability estimate given by the formula (48), and no restriction on the "size" of the analytic perturbation h_0, on $0 < \delta < T < \infty$, the electric field $E^\varepsilon(t, x)$ converges exponentially fast to zero in $L^\infty(\delta, T; L^2(\mathbb{T}^d))$ as $\varepsilon \to 0$.

Proof For strong convergence, one follows [17]. First introduce the function $F^\varepsilon(\tau, x, v)$ solution of the equations

$$\partial_\tau F^\varepsilon + v \cdot \nabla_x F^\varepsilon + E^\varepsilon [\![\varepsilon \rho^\varepsilon]\!] \cdot \nabla_v F^\varepsilon\,,$$

with

$$E^\varepsilon [\![\cdot]\!] = \nabla \Delta^{-1} \cdot\,, \quad f^\varepsilon(t, x, v) = F^\varepsilon(t/\varepsilon^2, x, v) \quad \text{, and} \quad \varepsilon E^\varepsilon[\rho^\varepsilon] = E^\varepsilon[\varepsilon \rho^\varepsilon]\,.$$

For $\varepsilon \leq \varepsilon_0$ small enough, apply Theorem 2 to the solution $(F_\varepsilon(\tau,x,v), E_\varepsilon(\tau,x))$ of

$$\begin{aligned}&\partial_\tau F + v\cdot\nabla_x F + E(\tau)\cdot\nabla_v F = 0\,,\\ &E = \nabla\Delta^{-1}\int_{\mathbb{R}^d_v} F dv - 1,\\ &F_\varepsilon(0,x,v) = G(v) + \varepsilon h_0(x,v)\,,\end{aligned}$$

which coincide with the solution $(f^\varepsilon(t,x,v), E^\varepsilon(t,x))$ of (51) through the relation $(f^\varepsilon(t,x,v), E^\varepsilon(t,x,v)) = (F_\varepsilon(t/\varepsilon^2,x,v), E_\varepsilon(t/\varepsilon^2,x))$. □

3.4 Roadmap for the Short-Time Quasilinear Approximation

In this section, we present a prospective method to prove the validity of a short-time quasilinear approximation in the presence of unstable eigenvalues. Observe that for any time $t > 0$ with

$$f^\varepsilon(t,x,v) = G(t,v) + \varepsilon h(t,x,v)\,,\qquad \fint h(t,x,v)dx = 0\,,$$

the Vlasov–Poisson equations are equivalent to the system

$$\begin{aligned}&\partial_t G + \varepsilon^2\nabla_v\cdot\Big(\fint E[h]h dx\Big) = 0\,,\quad E[h] = \nabla\Delta^{-1}\int_{\mathbb{R}^d_v} h(t,x,v)dv\,,\\ &\partial_t h + v\cdot\nabla_x h + E[h]\cdot\nabla_v G = -\varepsilon\nabla_v\cdot\Big(E[h]h - \fint E[h]hdx\Big)\,.\end{aligned}\tag{52}$$

Next, assume the existence of a simple non-degenerated root $(\Re\lambda(0) > 0, k)$ of the dispersion equation

$$1 - \frac{1}{|k|^2}\int_{\mathbb{R}^d_v}\frac{ik\cdot\nabla_v G(0,v)}{\lambda + ik\cdot v}dv = 0\,,\tag{53}$$

with $G(0,v) = G_0(v)$, and consider solutions $f^\varepsilon(t,x,v)$ of the Vlasov equation with complex initial data

$$f^\varepsilon(0,x,v) = G_0(v) + \varepsilon h_0(x,v) = G_0(v) + \varepsilon\frac{E(0,k)\cdot\nabla_v G_0(v)}{\lambda + ik\cdot v}e^{ik\cdot x}\,.$$

Assuming that $f^\varepsilon(0,x,v)$ is analytic, one observes (as proven by Benachour in [2]) that the corresponding solution of the Vlasov equation is also analytic. Hence (see

[19] Chapter 2, Section 1), the root can be extended on a finite time interval as simple solution of the equation

$$1 - \frac{1}{|k|^2} \int_{\mathbb{R}^d_v} \frac{ik \cdot \nabla_v G(t, v)}{\lambda(t) + ik \cdot v} dv = 0 ,$$

and then one introduces the approximate solution

$$\tilde{h}(t, x, v) = \frac{E(0, k) \cdot \nabla_v G(t, v)}{\lambda(t) + ik \cdot v} e^{\int_0^t ds \lambda(s) + ik \cdot x} . \tag{54}$$

The function $\tilde{h}$ will be used to construct an approximate diffusion $\widetilde{\mathbb{D}}^\varepsilon(v)$ such that, for short time, one has

$$\partial_t G(t, v) - \nabla_v \cdot \left(\widetilde{\mathbb{D}}^\varepsilon(v) \nabla_v G(t, v)\right) = O(\varepsilon^3),$$

while what follows from Eq. (52) is the estimate

$$\partial_t G(t, v) = O(\varepsilon^2) . \tag{55}$$

Since $\lambda(t)$ and of course $G(t)$ itself are analytic functions, from (55) and (54), one deduces

$$\partial_t \tilde{h} + v \cdot \nabla_x \tilde{h} + E[\tilde{h}] \cdot \nabla_v G(t) = O(\varepsilon^2) .$$

Hence, one also obtains

$$\partial_t (h - \tilde{h}) + v \cdot \nabla_x (h - \tilde{h}) + E[h - \tilde{h}] \cdot \nabla_v G = -\varepsilon \nabla_v \cdot \left(E[h]h - ⨍ E[h]h dx \right) + O(\varepsilon^2) .$$

Then, with $(h - \tilde{h})(0, x, v) = 0$, one obtains $h(t) - \tilde{h}(t) = O(\varepsilon)$, which eventually implies

$$\partial_t G + \varepsilon^2 \nabla_v \cdot \left(⨍ E[\tilde{h}]\tilde{h} dx \right) = \varepsilon^2 \nabla_v \cdot ⨍ \left(E[\tilde{h}]\tilde{h} dx - E[h]h \right) dx = O(\varepsilon^3) .$$

As pointed above, if (λ, k) are solutions of the dispersion equation, then the same is true for $(\lambda^\star, -k)$, and one can extend the above comparison between genuine solutions with real initial data

$$f^\varepsilon(0, x, v) = G_0(v) + \varepsilon \Re(h(0, x, v)) , \tag{56}$$

and the approximate solutions given by

$$\tilde{f}^\varepsilon(t, x, v) = G(t, v) + \varepsilon \Re \tilde{h}(t, x, v) .$$

Since the function $\tilde{h}$ satisfies the relation

$$(\lambda + ik \cdot v)\tilde{h}(t,k,v) + E[\tilde{h}(t,k,v)] \cdot \nabla_v G(t,v) = 0\,, \tag{57}$$

one obtains, for solutions with initial data given by (56),

$$\partial_t G(t,v) - \varepsilon^2 \nabla_v \cdot \left(\frac{E(0,k) \otimes (E(0,k))^\star \, \Re\lambda e^{2\Re \int_0^t ds \lambda(s)}}{(k \cdot v - \Im\lambda)^2 + (\Re\lambda)^2} \nabla_v G(t,v) \right)$$
$$= \partial_t G(t,v) + \varepsilon^2 \nabla_v \cdot \Re\left(\fint E[\tilde{h}]\tilde{h} dx \right) = O(\varepsilon^3)\,.$$

The above construction can be combined with the Dunford–Kato formula (47). Assuming again that the roots of the dispersion relation $(\lambda, k(\lambda))$ are simple, one obtains, for any initial data and any $\delta > 0$, summing with respect to the solutions of the dispersion equation (53) with $\Re\lambda > \delta$ the equation

$$\partial_t G(t,v) - \sum_{\Re\lambda > \delta} \varepsilon^2 \nabla_v \cdot \left(\frac{E(0,k(\lambda)) \otimes (E(0,k(\lambda)))^\star \, \Re\lambda e^{2\Re \int_0^t ds \lambda(s)}}{(k(\lambda) \cdot v - \Im\lambda)^2 + (\Re\lambda)^2} \nabla_v G(t,v) \right)$$
$$= O(\varepsilon^3) + O(\varepsilon^2) e^{\delta t}\,,$$

which is the standard quasilinear approximation.

4 Remarks and Conclusion

As said in the introduction, the thread in this contribution is the comparison from a genuine mathematical point of view of different approaches leading to the quasilinear diffusion approximation.

Remark 4 The most natural one is with the introduction of the rescaled equation. However, in such configuration, it has been shown that the stochastic scenario, which implies that the electric field is independent of the density, is almost compulsory. As such, one could also start from a stochastic flow solution of the ODEs

$$\frac{d}{dt} X^\varepsilon(t) = \frac{1}{\varepsilon^2} V^\varepsilon(t)\,,$$
$$\frac{d}{dt} V^\varepsilon(t) = -\frac{1}{\varepsilon} E^\varepsilon(t, X^\varepsilon(t))\,,$$

with a convenient correlation function A_τ leading directly at the macroscopic level, without the kinetic step in between, to a diffusion equation

$$\partial_t \overline{f^\varepsilon} - \nabla_v \cdot (\mathbb{D}(v)\nabla_v \overline{f^\varepsilon}) = 0\,.$$

Comparison with the diffusion matrix given also in terms of A_τ by (41) leads in such cases to a closed formula for the determination of such diffusion. For an interpretation at the level of plasma physics for such relation, see reference [1] and the original ones [11, 31]. Complete proofs may be obtained following the contributions [12, 14, 15, 20].

Weak convergence and introduction of randomness imply that analysis should be made on the solution rather than on the equation. This leads to the introduction of the Duhamel series that may be considered as an avatar of other BBGKY hierarchies. However, in [26, 28], the authors have introduced some decorrelation properties valid at any time, which close the Duhamel series at second order. This is the road that was followed in this contribution.

Remark 5 The short allusion to the Landau damping was motivated on one hand by the comparison with issue of strong, versus weak convergence, to zero of the electric field in the rescaled equation and on the other hand to underline the role of estimates on the charge density $\rho[h]$ under the stability condition (48), which allows to extend the resolvent beyond the imaginary axis.

Remark 6 The short-time validity of the quasilinear approximation is based on even much more formal presentations that can be found in basic plasma physics textbooks (see for instance [21] pages 514–517). Here, the symbol $O(\varepsilon)$, used everywhere, should be clarified for a complete proof, which will be the matter of a future work. For short time, systematic use of the Nash–Moser theorem should balance the loss derivative (of order 1 with respect to v) in Sect. 3.4. One should keep in mind the striking difference between the rescaled diffusion scenario with an independent and non-self-consistent stochastic vector field, which is the typical model for long-time dynamics, and the short-time diffusion scenario, where the self-consistent electric field is slaved to the solution by the Poisson equation. In fact, these two diffusion regimes are present in the nonlinear relaxation of the weak warm beam–plasma instability problem. Self-consistent numerical simulations of such problem [3] confirm the existence of these two diffusion regimes, plus a third regime between the two. In this third regime, dubbed the "trapping turbulent regime" in plasma physics literature, nonlinear wave–wave coupling plays an important role. Until now and up to our knowledge, there is no even a roadmap for a full mathematical description of such regime.

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